

High-energy amplitudes in $\mathcal{N} = 4$ SYM at the next-to-leading order

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JTI workshop April 14, 2009

- Regge limit in a conformal theory.
- High-energy scattering and Wilson lines.
- Evolution equation for color dipoles.
- Leading order: BK equation.
- Non-linear evolution equation in the NLO.
- $\mathcal{N} = 4$: study of 2-dim conformal invariance at high energies
- NLO BK kernel in $\mathcal{N} = 4$.
- NLO amplitude in $\mathcal{N} = 4$ SYM
- Conclusions.

Conformal four-point amplitude

$$A(x, y, x', y') = (x - y)^4 (x' - y')^4 N_c^2 \langle \mathcal{O}(x) \mathcal{O}^\dagger(y) \mathcal{O}(x') \mathcal{O}^\dagger(y') \rangle$$

$\mathcal{O} = \text{Tr}\{Z^2\}$ ($Z = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$) - chiral primary operator

In a conformal theory the amplitude is a function of two conformal ratios

$$\begin{aligned} A &= F(R, R') \\ R &= \frac{(x - y)^2 (x' - y')^2}{(x - x')^2 (y - y')^2} R' = \frac{(x - y)^2 (x' - y')^2}{(x - y')^2 (x' - y)^2} \end{aligned}$$

At large N_c

$$A(x, y, x', y') = A(g^2 N_c) \quad g^2 N_c = \lambda \text{ -- 't Hooft coupling}$$

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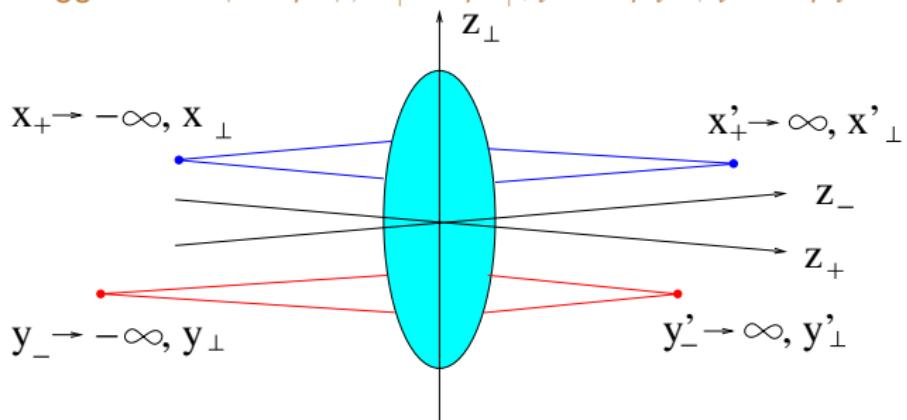
AdS/CFT gives predictions at large $\lambda \rightarrow \infty$.

Our goal is perturbative expansion and resummation of $(\lambda \ln s)^n$ at large energies in the next-to-leading approximation

$$(\lambda \ln s)^n (c_n^{\text{LO}} + c_n^{\text{NLO}} \lambda)$$

Regge limit in the coordinate space

Regge limit: $x_+ \rightarrow \rho x_+$, $x'_+ \rightarrow \rho x'_+$, $y_- \rightarrow \rho' y_-$, $y'_- \rightarrow \rho' y'_-$ $\rho, \rho' \rightarrow \infty$

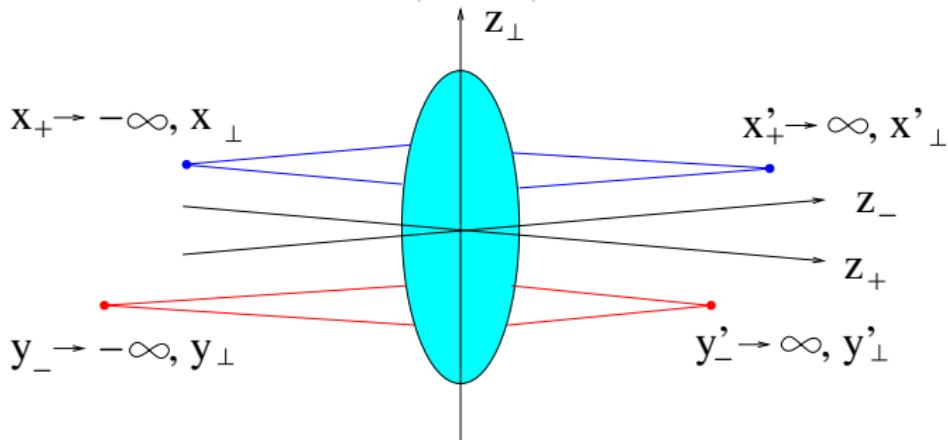


Full 4-dim conformal group: $A = F(R_1, R_2)$

$$\begin{aligned}
 R &= \frac{(x-y)^2(x'-y')^2}{(x-x')^2(y-y')^2} \rightarrow \frac{\rho^2\rho'^2 x_+ x'_+ y_- y'_-}{(x-x')_\perp^2 (y-y')_\perp^2} \\
 r &= \frac{[(x-y)^2(x'-y')^2 - (x'-y)^2(x-y')^2]^2}{(x-x')^2(y-y')^2(x-y)^2(x'-y')^2} \\
 &\rightarrow \frac{[(x'-y')_\perp^2 x_+ y_- + x'_+ y'_- (x-y)_\perp^2 + x_+ y'_- (x'-y)_\perp^2 + x'_+ y_- (x-y')_\perp^2]^2}{(x-x')_\perp^2 (y-y')_\perp^2 x_+ x'_+ y_- y'_-}
 \end{aligned}$$

4-dim conformal group versus $SL(2, C)$

Regge limit: $x_+ \rightarrow \rho x_+$, $x'_+ \rightarrow \rho x'_+$, $y_- \rightarrow \rho' y_-$, $y'_- \rightarrow \rho' y'_-$ $\rho, \rho' \rightarrow \infty$



Regge limit symmetry: 2-dim conformal group $SL(2, C)$ formed from P_1, P_2, M^{12}, D, K_1 and K_2 which leave the plane $(0, 0, z_\perp)$ invariant.

Pomeron in a conformal theory

$$A(x, y; x', y') \stackrel{s \rightarrow \infty}{=} \frac{i}{2} \int d\nu f_+(\omega(\lambda, \nu)) F(\lambda, \nu) \Omega(r, \nu) R^{\omega(\lambda, \nu)/2}$$

L. Cornalba (2007)

$f_+(\omega) = \frac{e^{i\pi\omega} - 1}{\sin \pi\omega}$ - signature factor

$\Omega(r, \nu)$ - solution of the eqn $(\square_{H_3} + \nu^2 + 1)\Omega(r, \nu) = 0$.

Explicit form:

$$\Omega(r, \nu) = \frac{\nu^2}{\pi^3} \int d^2 z \left(\frac{\kappa^2}{(2\kappa \cdot \zeta)^2} \right)^{\frac{1}{2} + i\nu} \left(\frac{\kappa'^2}{(2\kappa' \cdot \zeta)^2} \right)^{\frac{1}{2} - i\nu}$$

$$\zeta = p_1 + \frac{z_\perp^2}{s} p_2 + z_\perp, \quad p_1^2 = p_2^2 = 0, \quad 2(p_1, p_2) = s$$

$$\kappa = \frac{1}{2x_+}(p_1 - \frac{x^2}{s}p_2 + x_\perp) - \frac{1}{2y_+}(p_1 - \frac{y^2}{s}p_2 + y_\perp), \quad \kappa^2 \kappa'^2 = \frac{1}{R}$$

$$\kappa' = \frac{1}{2x'_-}(p_1 - \frac{x'^2}{s}p_2 + x'_\perp) - \frac{1}{2y'_-}(p_1 - \frac{y'^2}{s}p_2 + y'_\perp), \quad 4(\kappa \cdot \kappa')^2 = \frac{r}{R}$$

The dynamics is described by $\omega(\lambda, \nu)$ and $F(\lambda, \nu)$.

Pomeron in the conformal theory

$$A(x, y; x', y') \stackrel{s \rightarrow \infty}{\equiv} \frac{i}{2} \int d\nu f_+(\omega(\lambda, \nu)) F(\lambda, \nu) \Omega(r, \nu) R^{\omega(\lambda, \nu)/2}$$

Pomeron intercept $\omega(\nu, \lambda)$ is known in two limits:

1. $\lambda \rightarrow 0 :$ $\omega(\nu, \lambda) = \frac{\lambda}{\pi} \chi(\nu) + \lambda^2 \omega_1(\nu) + \dots$

$\chi(\nu) = 2\psi(1) - \psi(\tfrac{1}{2} + i\nu) - \psi(\tfrac{1}{2} - i\nu)$ - BFKL intercept,

$\omega_1(\nu)$ - NLO BFKL intercept Lipatov, Kotikov (2000)

2. $\lambda \rightarrow \infty :$ $AdS/CFT \Rightarrow \omega(\nu, \lambda) = 2 - \frac{\nu^2 + 4}{2\sqrt{\lambda}} + \dots$

2 = graviton spin , next term - Brower, Polchinski, Strassler, Tan (2006)
Cornalba, Costa, Penedones (2007)

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The function $F(\nu, \lambda)$ in two limits:

1. $\lambda \rightarrow 0 :$ $F(\nu, \lambda) = \lambda^2 F_0(\nu) + \lambda^3 F_1(\nu) + \dots$

$$F_0(\nu) = \frac{\pi \sinh \pi \nu}{4\nu \cosh^3 \pi \nu} \quad \text{Cornalba, Costa, Penedones (2007)}$$

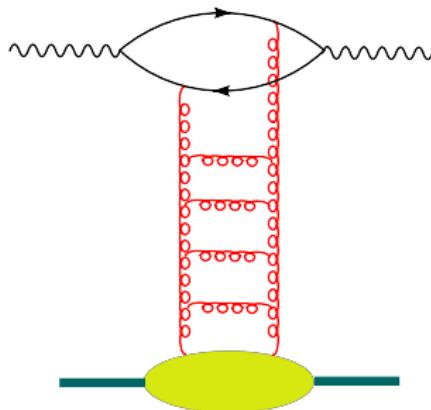
$$F_1(\nu) = \text{see below} \quad \text{G. Chirilli and I.B. (2009)}$$

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L.Cornalba (2007)

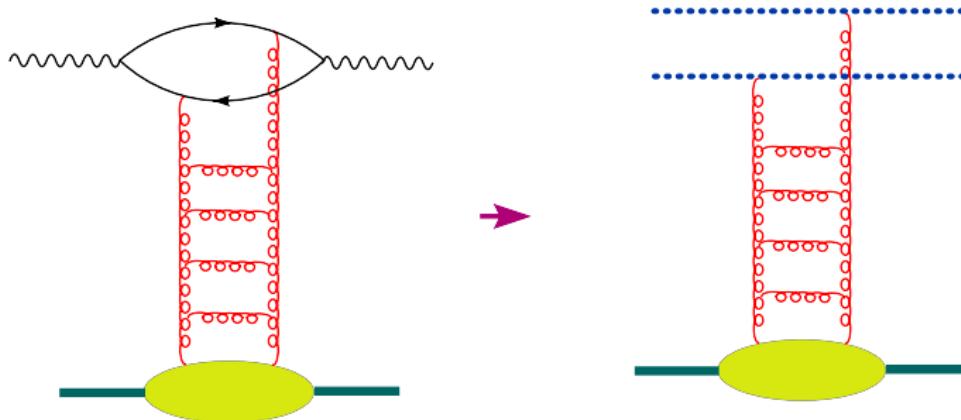
We calculate $F_1(\nu)$ (and confirm $\omega_1(\nu)$) using the expansion of high-energy amplitudes in Wilson lines (color dipoles)

- At high energies, particles move along straight lines \Rightarrow the amplitude of $\gamma^* A \rightarrow \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):



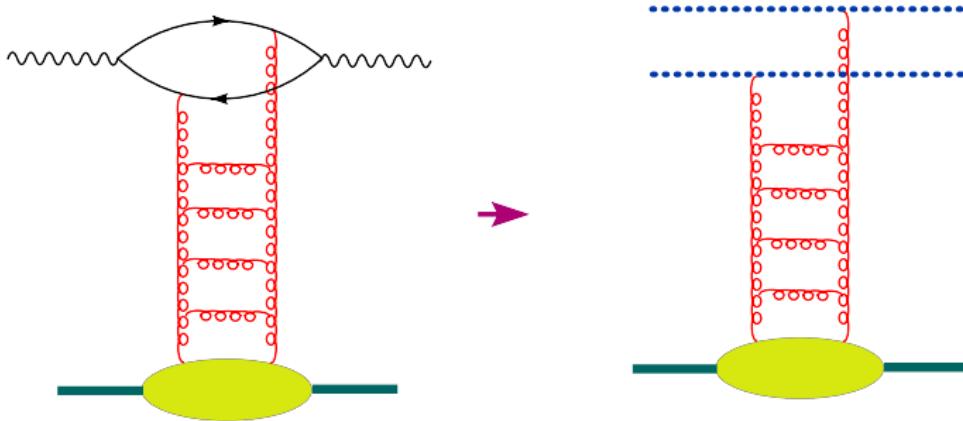
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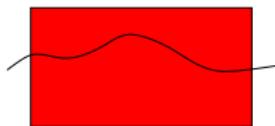
$$A(s) = \int \frac{d^2 k_\perp}{4\pi^2} I^A(k_\perp) \langle B | \text{Tr}\{ U(k_\perp) U^\dagger(-k_\perp) \} | B \rangle$$

$$U(x_\perp) = P e^{ig \int_{-\infty}^{\infty} du n^\mu A_\mu(u n + x_\perp)}$$

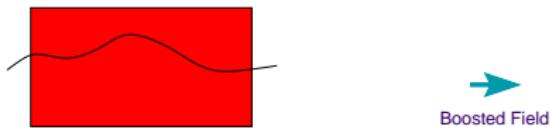
Wilson line

Spectator frame: propagation in the shock-wave background.

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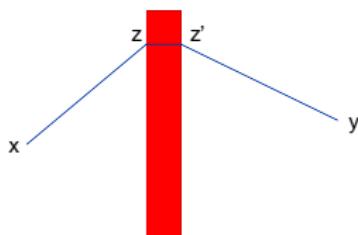


Each path is weighted with the gauge factor $P e^{ig \int dx_\mu A^\mu}$. Quarks and gluons do not have time to deviate in the transverse space \Rightarrow we can replace the gauge factor along the actual path with the one along the straight-line path.

Spectator frame: propagation in the shock-wave background.



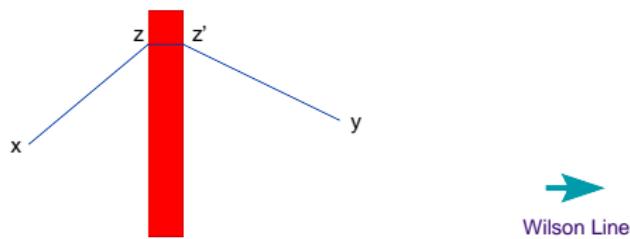
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$[x \rightarrow z: \text{free propagation}] \times$

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[$x \rightarrow z$: free propagation] \times
[$U^{ab}(z_\perp)$ - instantaneous interaction with the $\eta < \eta_2$ shock wave] \times

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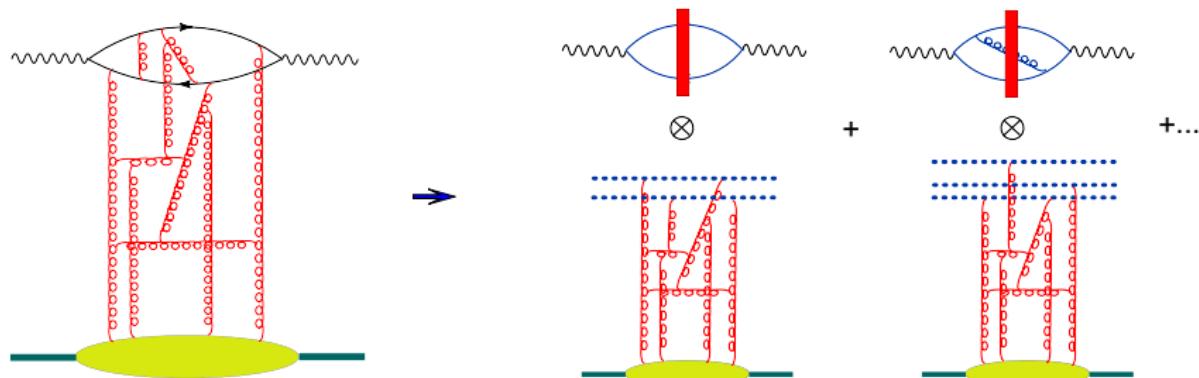


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[$z \rightarrow y$: free propagation]

Expansion of the amplitude in color dipoles in the NLO



η - rapidity factorization scale

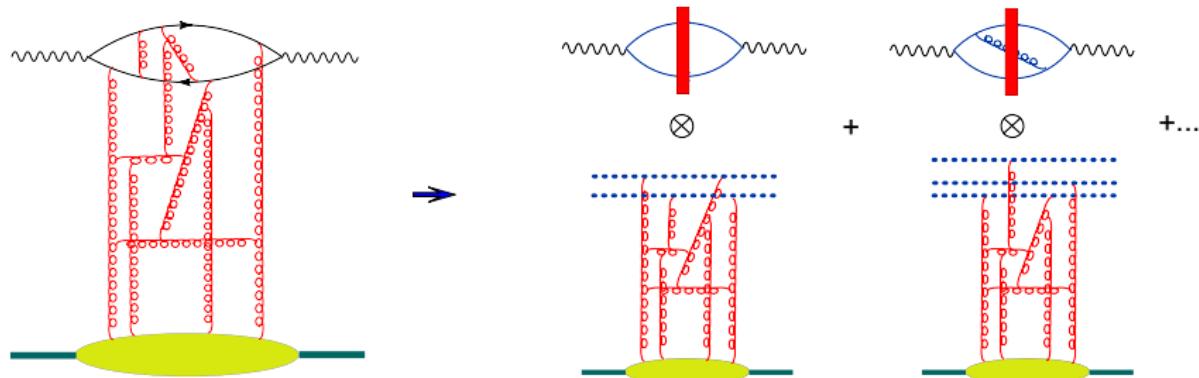
Rapidity $> \eta$ - coefficient function ("impact factor")

Rapidity $< \eta$ - matrix elements of (light-like) Wilson lines with rapidity divergence cut by η

$$U_x^\eta = \text{Pexp} \left[ig \int_{-\infty}^{\infty} du p_1^\mu A_\mu^\eta (up_1 + x_\perp) \right]$$

$$A_\mu^\eta(x) = \int \frac{d^4 k}{(2\pi)^4} \theta(e^\eta - |\alpha_k|) e^{-ik \cdot x} A_\mu(k)$$

Expansion of the amplitude in color dipoles in the NLO



The high-energy operator expansion is

$$T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \int d^2 z_1 d^2 z_2 I^{\text{LO}}(z_1, z_2) \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}$$

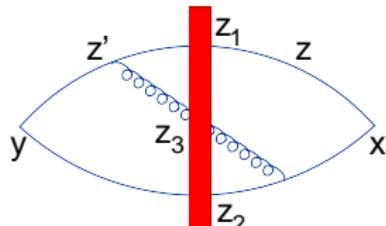
$$+ \int d^2 z_1 d^2 z_2 d^2 z_3 I^{\text{NLO}}(z_1, z_2, z_3) [\frac{1}{N_c} \text{Tr}\{T^n \hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta} T^n \hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]$$

In the leading order - conf. invariant impact factor

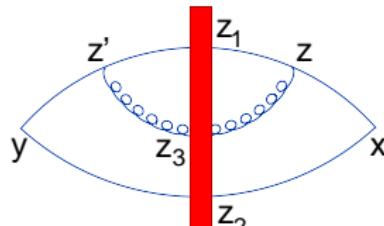
$$I_{\text{LO}} = \frac{x_+^{-2} y_+^{-2}}{\pi^2 \mathcal{Z}_1^2 \mathcal{Z}_2^2}, \quad \mathcal{Z}_i \equiv \frac{(x - z_i)_\perp^2}{x_+} - \frac{(y - z_i)_\perp^2}{y_+}$$

CCP, 2007

NLO impact factor



(a)



(b)

$$I^{\text{NLO}}(x, y; z_1, z_2, z_3; \eta) = -I^{\text{LO}} \times \frac{\lambda}{\pi^2} \frac{z_{13}^2}{z_{12}^2 z_{23}^2} \left[\ln \frac{\sigma s}{4} \mathcal{Z}_3 - \frac{i\pi}{2} + C \right]$$

The NLO impact factor is not Möbius invariant \Rightarrow the color dipole with the cutoff η is not invariant

However, if we define a composite operator (a - analog of μ^{-2} for usual OPE)

$$\begin{aligned} [\text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]^{\text{conf}} &= \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \\ &+ \frac{\lambda}{2\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} [\text{Tr}\{T^n \hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta} T^n \hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - N_c \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}] \ln \frac{az_{12}^2}{z_{13}^2 z_{23}^2} + O(\lambda^2) \end{aligned}$$

the impact factor becomes conformal in the NLO.

Operator expansion in conformal dipoles

$$T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \int d^2z_1 d^2z_2 I^{\text{LO}}(z_1, z_2) \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}^{\text{conf}}$$
$$+ \int d^2z_1 d^2z_2 d^2z_3 I^{\text{NLO}}(z_1, z_2, z_3) \left[\frac{1}{N_c} \text{Tr}\{T^n \hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta} T^n \hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \right]$$
$$I^{\text{NLO}} = -I^{\text{LO}} \frac{\lambda}{2\pi^2} \int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[\ln \frac{z_{12}^2 e^{2\eta} a s^2}{z_{13}^2 z_{23}^2} \mathcal{Z}_3^2 - i\pi + 2C \right]$$

The new NLO impact factor is conformally invariant

$\Rightarrow \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}^{\text{conf}}$ is Möbius invariant

We think that one can construct the composite conformal dipole operator order by order in perturbation theory.

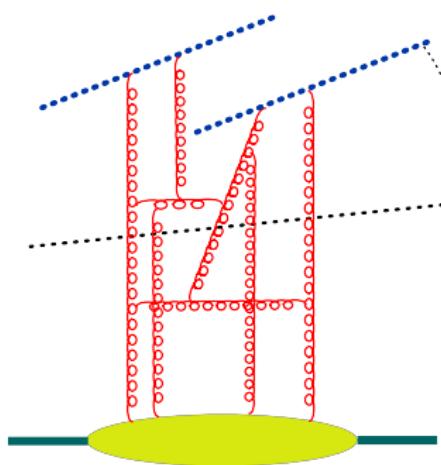
Analogy: when the UV cutoff does not respect the symmetry of a local operator, the composite local renormalized operator in must be corrected by finite counterterms order by order in perturbation theory.

Evolution equation for color dipoles

To get the evolution equation, consider the dipole with the rapidities up to η_1 and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to η_2).

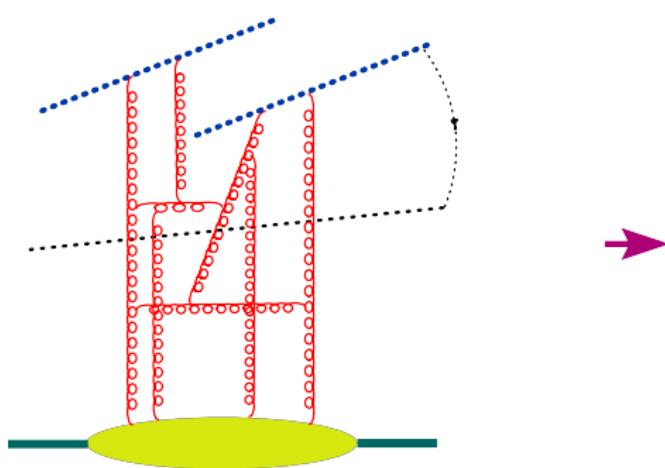
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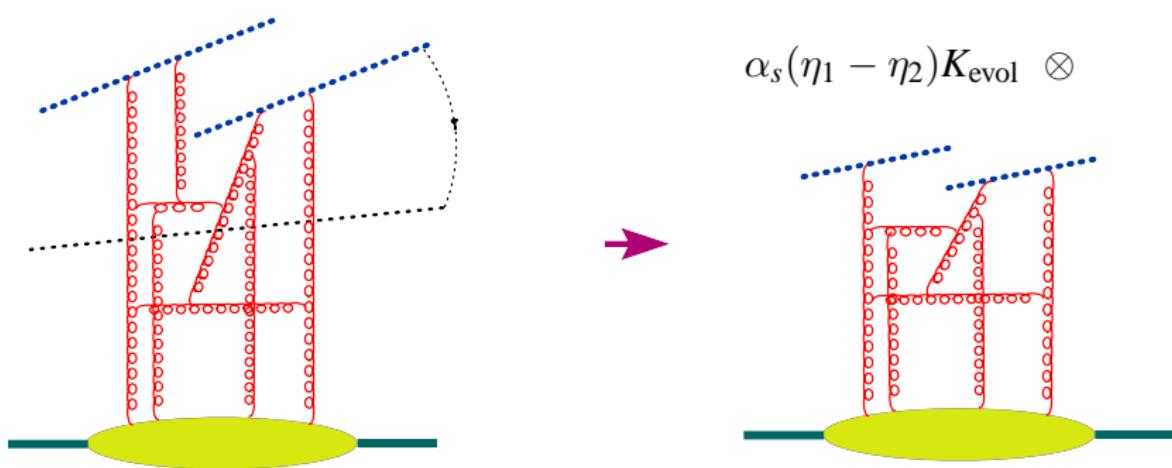
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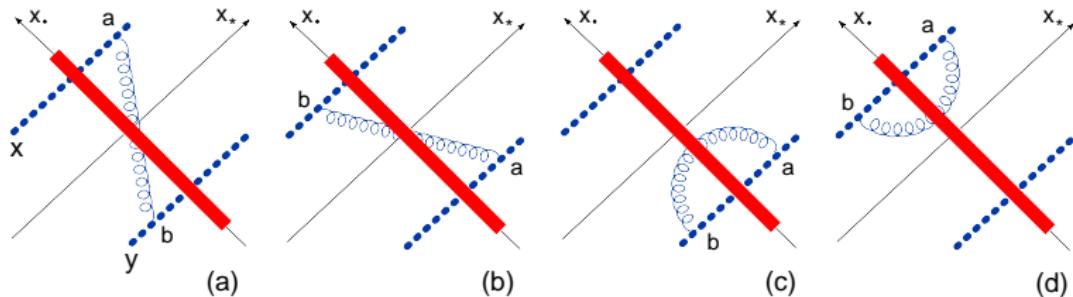
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Leading order: BK equation

$$\frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \dots \Rightarrow$$

$$\frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}} = \langle K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}}$$



$$U_z^{ab} = \text{Tr}\{t^a U_z t^b U_z^\dagger\} \Rightarrow (U_x U_y^\dagger)^{\eta_1} \rightarrow (U_x U_y^\dagger)^{\eta_1} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

\Rightarrow Evolution equation is non-linear

Non linear evolution equation

$$\hat{\mathcal{U}}(x, y) \equiv 1 - \frac{1}{N_c} \text{Tr}\{\hat{U}(x_\perp) \hat{U}^\dagger(y_\perp)\}$$

Non linear evolution equation

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BK equation

$$\frac{d}{d\eta} \hat{\mathcal{U}}(x, y) = \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2 z}{(x-z)^2 (y-z)^2} \left\{ \hat{\mathcal{U}}(x, z) + \hat{\mathcal{U}}(z, y) - \hat{\mathcal{U}}(x, y) - \hat{\mathcal{U}}(x, z) \hat{\mathcal{U}}(z, y) \right\}$$

I. B. (1996), Yu. Kovchegov (1999)

Alternative approach: JIMWLK (1997-2000)

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Alternative approach: JIMWLK (1997-2000)

LLA for DIS in pQCD \Rightarrow BFKL

(LLA: $\alpha_s \ll 1, \alpha_s \eta \sim 1$)

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Alternative approach: JIMWLK (1997-2000)

LLA for DIS in pQCD \Rightarrow BFKL

(LLA: $\alpha_s \ll 1, \alpha_s \eta \sim 1$)

LLA for DIS in sQCD \Rightarrow BK eqn

(LLA: $\alpha_s \ll 1, \alpha_s \eta \sim 1, \alpha_s A^{1/3} \sim 1$)

(s for semiclassical)

Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] = \text{Pexp} \left\{ ig \int_{-\infty}^{\infty} dx^+ A_+(x^+, x_\perp) \right\}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

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Indeed,

$(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after the inversion $x_\perp \rightarrow x_\perp/x_\perp^2$ and $x^+ \rightarrow x^+/x_\perp^2$

Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] = \text{Pexp} \left\{ ig \int_{-\infty}^{\infty} dx^+ A_+(x^+, x_\perp) \right\}$$

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\Rightarrow The dipole kernel is invariant under the inversion $V(x_\perp) = U(x_\perp/x_\perp^2)$

$$\frac{d}{d\eta} \text{Tr}\{V_x V_y^\dagger\} = \frac{\alpha_s}{2\pi^2} \int \frac{d^2 z}{z^4} \frac{(x-y)^2}{(x-z)^2(z-y)^2} [\text{Tr}\{V_x V_z^\dagger\} \text{Tr}\{V_z V_y^\dagger\} - N_c \text{Tr}\{V_x V_y^\dagger\}]$$

Conformal invariance of the BK equation

SL(2,C) for Wilson lines

$$\hat{S}_- \equiv \frac{i}{2}(K^1 + iK^2), \quad \hat{S}_0 \equiv \frac{i}{2}(D + iM^{12}), \quad \hat{S}_+ \equiv \frac{i}{2}(P^1 - iP^2)$$

$$[\hat{S}_0, \hat{S}_\pm] = \pm \hat{S}_\pm, \quad \frac{1}{2}[\hat{S}_+, \hat{S}_-] = \hat{S}_0,$$

$$[\hat{S}_-, \hat{U}(z, \bar{z})] = z^2 \partial_z \hat{U}(z, \bar{z}), \quad [\hat{S}_0, \hat{U}(z, \bar{z})] = z \partial_z \hat{U}(z, \bar{z}), \quad [\hat{S}_+, \hat{U}(z, \bar{z})] = -\partial_z \hat{U}(z, \bar{z})$$

$$z \equiv z^1 + iz^2, \bar{z} \equiv z^1 - iz^2, \quad U(z_\perp) = U(z, \bar{z})$$

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Conformal invariance of the evolution kernel

$$\begin{aligned} \frac{d}{d\eta} [\hat{S}_-, \text{Tr}\{U_x U_y^\dagger\}] &= \frac{\alpha_s N_c}{2\pi^2} \int dz K(x, y, z) [\hat{S}_-, \text{Tr}\{U_x U_y^\dagger\} \text{Tr}\{U_x U_y^\dagger\}] \\ &\Rightarrow \left[x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + z^2 \frac{\partial}{\partial z} \right] K(x, y, z) = 0 \end{aligned}$$

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In the leading order - OK. In the NLO - ?

Non-linear evolution equation in the NLO

$$\begin{aligned} \frac{d}{d\eta} Tr\{U_x U_y^\dagger\} = \\ \int \frac{d^2 z}{2\pi^2} \left(\alpha_s \frac{(x-y)^2}{(x-z)^2(z-y)^2} + \alpha_s^2 K_{NLO}(x,y,z) \right) [Tr\{U_x U_z^\dagger\} Tr\{U_z U_y^\dagger\} - N_c Tr\{U_z U_y^\dagger\}] + \\ \alpha_s^2 \int d^2 z d^2 z' \left(K_4(x,y,z,z') \{U_x, U_{z'}^\dagger, U_z, U_y^\dagger\} + K_6(x,y,z,z') \{U_x, U_{z'}^\dagger, U_{z'}, U_z, U_z^\dagger, U_y^\dagger\} \right) \end{aligned}$$

K_{NLO} is the next-to-leading order correction to the dipole kernel and K_4 and K_6 are the coefficients in front of the (tree) four- and six-Wilson line operators with arbitrary white arrangements of color indices.

Definition of the NLO kernel

In general

$$\frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + O(\alpha_s^3)$$

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$$\alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = \frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} - \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + O(\alpha_s^3)$$

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We calculate the “matrix element” of the r.h.s. in the shock-wave background

$$\langle \alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle = \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle - \langle \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle + O(\alpha_s^3)$$

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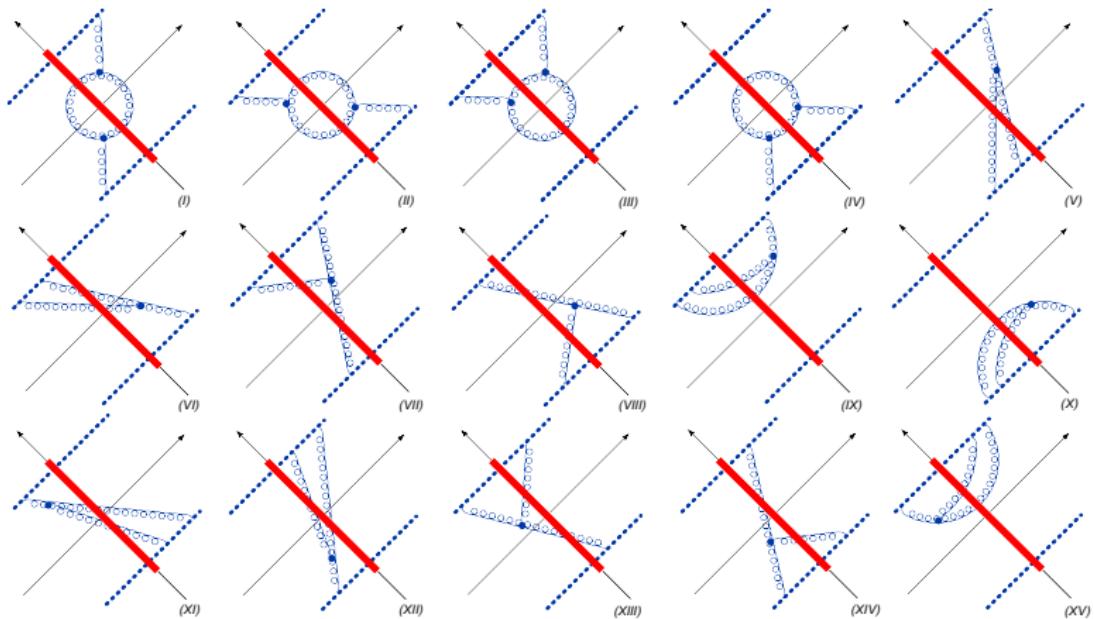
$$\langle \alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle = \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle - \langle \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle + O(\alpha_s^3)$$

Subtraction of the (LO) contribution (with the rigid rapidity cutoff)
⇒ $\left[\frac{1}{v}\right]_+$ prescription in the integrals over Feynman parameter v

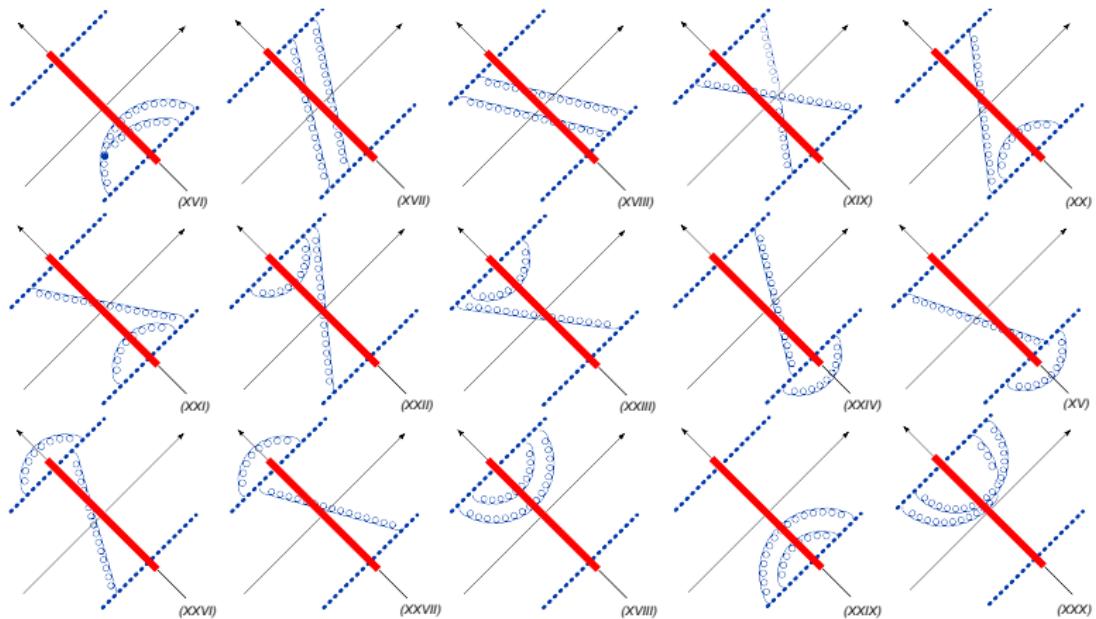
Typical integral

$$\int_0^1 dv \frac{1}{(k-p)_\perp^2 v + p_\perp^2 (1-v)} \left[\frac{1}{v}\right]_+ = \frac{1}{p_\perp^2} \ln \frac{(k-p)_\perp^2}{p_\perp^2}$$

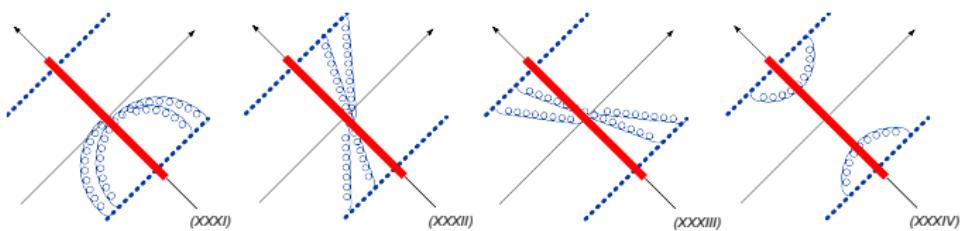
Gluon part of the NLO BK kernel: diagrams



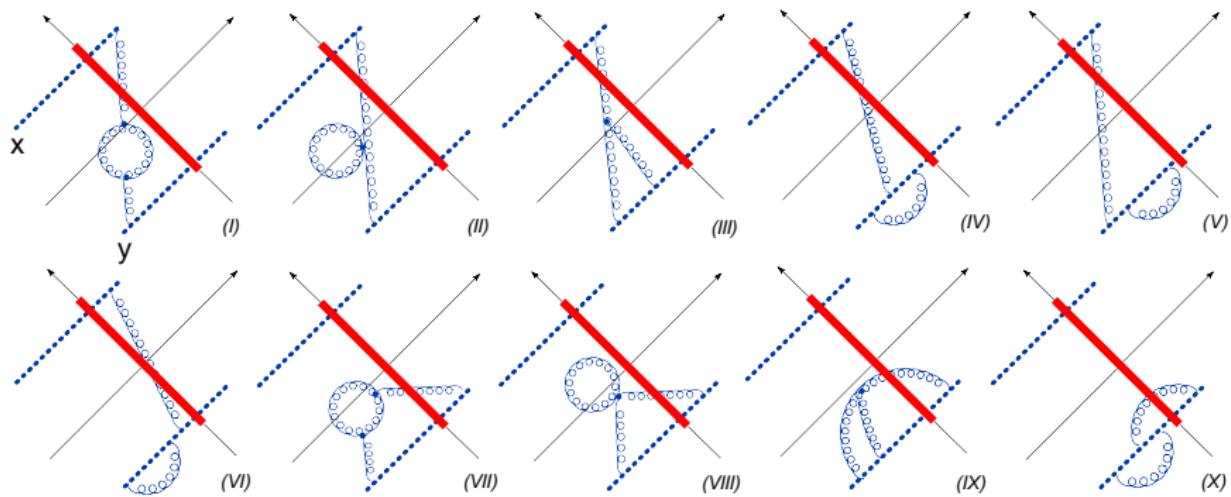
Diagrams for $1 \rightarrow 3$ dipoles transition



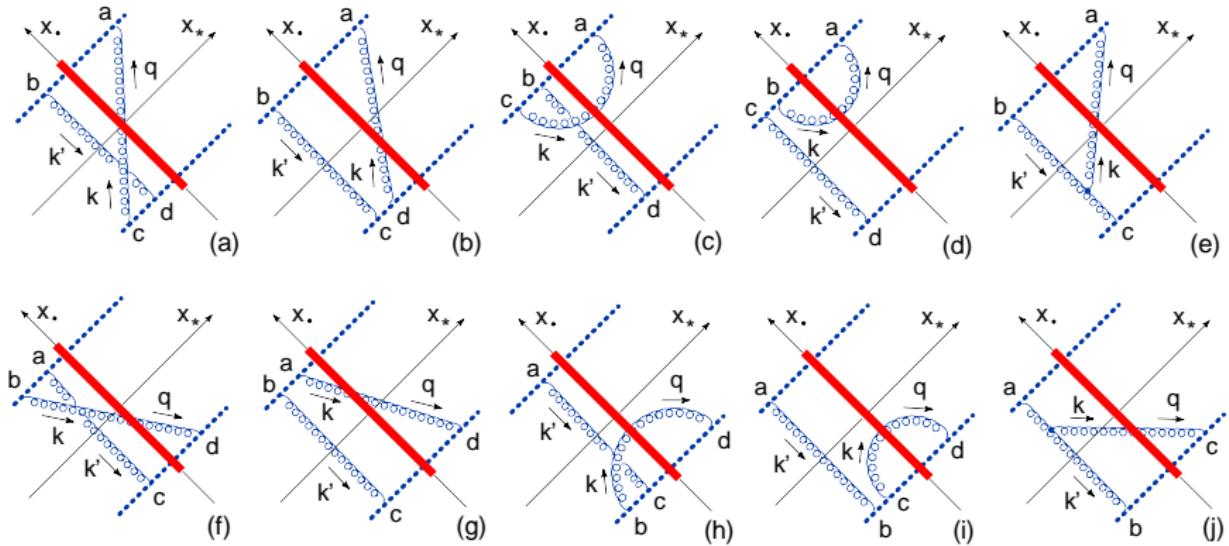
Diagrams for $1 \rightarrow 3$ dipoles transition



"Running coupling" diagrams

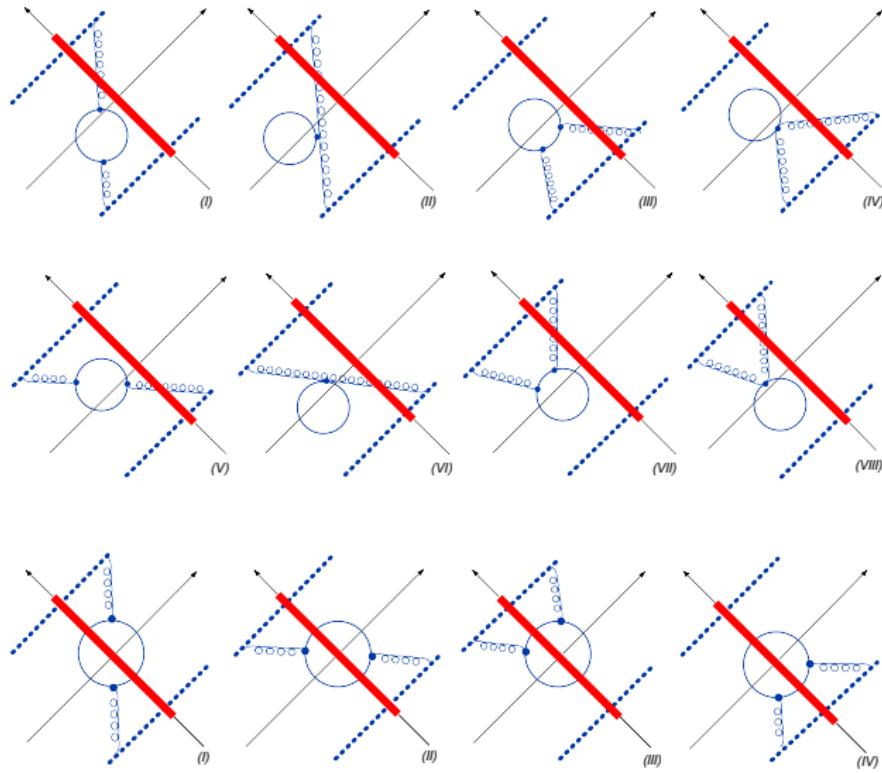


$1 \rightarrow 2$ dipole transition diagrams



$$\begin{aligned}
\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} = & \frac{\alpha_s}{2\pi^2} \int d^2 z \left([\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \right. \\
& \times \left\{ \frac{(x-y)^2}{X^2 Y^2} \left[1 + \frac{\alpha_s N_c}{4\pi} \left(\frac{11}{3} \ln(x-y)^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right) \right] \right. \\
& - \frac{11}{3} \frac{\alpha_s N_c}{4\pi} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} - \frac{\alpha_s N_c}{2\pi} \frac{(x-y)^2}{X^2 Y^2} \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \Big\} \\
& + \frac{\alpha_s}{4\pi^2} \int d^2 z' \left\{ [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \right. \\
& - (z' \rightarrow z)] \frac{1}{(z-z')^4} \left[-2 + \frac{X'^2 Y^2 + Y'^2 X^2 - 4(x-y)^2(z-z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right] \\
& + [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z' \rightarrow z)] \\
& \times \left. \left[\frac{(x-y)^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x-y)^2}{(z-z')^2 X^2 Y'^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right\}
\end{aligned}$$

$\mathcal{N} = 4$ diagrams (scalar and gluino loops)



$$\begin{aligned}
& \frac{d}{d\eta} \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \\
&= \frac{\alpha_s}{\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left\{ 1 - \frac{\alpha_s N_c}{4\pi} \left[\frac{\pi^2}{3} + 2 \ln \frac{z_{13}^2}{z_{12}^2} \ln \frac{z_{23}^2}{z_{12}^2} \right] \right\} \\
&\times [\text{Tr}\{T^a \hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta} T^a \hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - N_c \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}] \\
&- \frac{\alpha_s^2}{4\pi^4} \int \frac{d^2 z_3 d^2 z_4}{z_{34}^4} \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} \left[1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{23}^2 z_{14}^2} \right] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \\
&\times \text{Tr}\{[T^a, T^b] \hat{U}_{z_1}^\eta T^{a'} T^{b'} \hat{U}_{z_2}^{\dagger\eta} + T^b T^a \hat{U}_{z_1}^\eta [T^{b'}, T^{a'}] \hat{U}_{z_2}^{\dagger\eta}\} (\hat{U}_{z_3}^\eta)^{aa'} (\hat{U}_{z_4}^\eta - \hat{U}_{z_3}^\eta)^{bb'}
\end{aligned}$$

NLO kernel = Non-conformal term + Conformal analytic term.

Non-conformal term is due to the non-invariant cutoff $\alpha < \sigma = e^{2\eta}$ in the rapidity of Wilson lines.

$$\begin{aligned}
 & \frac{d}{d\eta} \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \\
 &= \frac{\alpha_s}{\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left\{ 1 - \frac{\alpha_s N_c}{4\pi} \left[\frac{\pi^2}{3} + 2 \ln \frac{z_{13}^2}{z_{12}^2} \ln \frac{z_{23}^2}{z_{12}^2} \right] \right\} \\
 &\quad \times [\text{Tr}\{T^a \hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta} T^a \hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - N_c \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}] \\
 &\quad - \frac{\alpha_s^2}{4\pi^4} \int \frac{d^2 z_3 d^2 z_4}{z_{34}^4} \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} \left[1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{23}^2 z_{14}^2} \right] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \\
 &\quad \times \text{Tr}\{[T^a, T^b] \hat{U}_{z_1}^\eta T^{a'} T^{b'} \hat{U}_{z_2}^{\dagger\eta} + T^b T^a \hat{U}_{z_1}^\eta [T^{b'}, T^{a'}] \hat{U}_{z_2}^{\dagger\eta}\} (\hat{U}_{z_3}^\eta)^{aa'} (\hat{U}_{z_4}^\eta - \hat{U}_{z_3}^\eta)^{bb'}
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For the conformal composite dipole the result is Möbius invariant

Evolution equation for composite conformal dipoles in $\mathcal{N} = 4$

$$\begin{aligned} & \frac{d}{d\eta} [\text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]^{\text{conf}} \\ &= \frac{\alpha_s}{\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[1 - \frac{\alpha_s N_c}{4\pi} \frac{\pi^2}{3} \right] [\text{Tr}\{T^a \hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta} T^a \hat{U}_{z_3} \hat{U}_{z_2}^{\dagger\eta}\} - N_c \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]^{\text{conf}} \\ & - \frac{\alpha_s^2}{4\pi^4} \int d^2 z_3 d^2 z_4 \frac{z_{12}^2}{z_{13}^2 z_{24}^2 z_{34}^2} \left\{ 2 \ln \frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2} + \left[1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2} \right] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \right\} \\ & \times \text{Tr}\{[T^a, T^b] \hat{U}_{z_1}^\eta T^{a'} T^{b'} \hat{U}_{z_2}^{\dagger\eta} + T^b T^a \hat{U}_{z_1}^\eta [T^{b'}, T^{a'}] \hat{U}_{z_2}^{\dagger\eta}\} [(\hat{U}_{z_3}^\eta)^{aa'} (\hat{U}_{z_4}^\eta)^{bb'} - (z_4 \rightarrow z_3)] \end{aligned}$$

Now Möbius invariant!

NLO BFKL equation in $\mathcal{N} = 4$ SYM

To find $A(x, y; x', y')$ we need the linearized (NLO BFKL) equation. With two-gluon accuracy

$$\hat{\mathcal{U}}^\eta(x, y) = 1 - \frac{1}{N_c^2 - 1} \text{Tr}\{\hat{U}_x^\eta \hat{U}_y^{\dagger\eta}\}$$

Conformal dipole operator in the BFKL approximation

$$\hat{\mathcal{U}}_{\text{conf}}^\eta(z_1, z_2) = \hat{\mathcal{U}}^\eta(z_1, z_2) + \frac{\alpha_s N_c}{4\pi^2} \int d^2 z \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \ln \frac{az_{12}^2}{z_{13}^2 z_{23}^2} [\hat{\mathcal{U}}^\eta(z_1, z_3) + \hat{\mathcal{U}}^\eta(z_2, z_3) - \hat{\mathcal{U}}^\eta(z_1, z_2)]$$

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To find $A(x, y; x', y')$ we need the linearized (NLO BFKL) equation. With two-gluon accuracy

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NLO BFKL

$$\begin{aligned} & \frac{d}{d\eta} \hat{\mathcal{U}}_{\text{conf}}^\eta(z_1, z_2) \\ &= \frac{\alpha_s N_c}{2\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[1 - \frac{\alpha_s N_c}{4\pi} \frac{\pi^2}{3} \right] [\hat{\mathcal{U}}_{\text{conf}}^\eta(z_1, z_3) + \hat{\mathcal{U}}_{\text{conf}}^\eta(z_2, z_3) - \hat{\mathcal{U}}_{\text{conf}}^\eta(z_1, z_2)] \\ &+ \frac{\alpha_s^2 N_c^2}{8\pi^4} \int \frac{d^2 z_3 d^2 z_4}{z_{34}^4} \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} \left\{ 2 \ln \frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2} + \left[1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2} \right] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \right\} \hat{\mathcal{U}}_{\text{conf}}^\eta(z_3, z_4) \\ &\quad + \frac{3\alpha_s^2 N_c^2}{2\pi^3} \zeta(3) \hat{\mathcal{U}}_{\text{conf}}^\eta(z_1, z_2) \end{aligned}$$

Pomeron intercept

Pomeron intercept = the eigenvalue of the NLO BFKL equation

$$\omega(n, \nu) = \frac{\alpha_s}{\pi} N_c \left[\chi(n, \frac{1}{2} + i\nu) + \frac{\alpha_s N_c}{4\pi} \delta(n, \frac{1}{2} + i\nu) \right],$$

$$\delta(n, \gamma) = 6\zeta(3) - \frac{\pi^2}{3} \chi(n, \gamma) - \chi''(n, \gamma) - 2\Phi(n, \gamma) - 2\Phi(n, 1 - \gamma)$$

where

$$\chi(n, \gamma) = 2\psi(1) - \psi(\gamma + \frac{n}{2}) - \psi(1 - \gamma + \frac{n}{2})$$

$$\begin{aligned} \Phi(n, \gamma) &= \int_0^1 \frac{dt}{1+t} t^{\gamma-1+\frac{n}{2}} \left\{ \frac{\pi^2}{12} - \frac{1}{2} \psi' \left(\frac{n+1}{2} \right) - \text{Li}_2(t) - \text{Li}_2(-t) \right. \\ &\quad \left. - \left(\psi(n+1) - \psi(1) + \ln(1+t) + \sum_{k=1}^{\infty} \frac{(-t)^k}{k+n} \right) \ln t - \sum_{k=1}^{\infty} \frac{t^k}{(k+n)^2} [1 - (-1)^k] \right\} \end{aligned}$$

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Pomeron intercept = the eigenvalue of the NLO BFKL equation

$$\omega(n, \nu) = \frac{\alpha_s}{\pi} N_c \left[\chi(n, \frac{1}{2} + i\nu) + \frac{\alpha_s N_c}{4\pi} \delta(n, \frac{1}{2} + i\nu) \right],$$

$$\delta(n, \gamma) = 6\zeta(3) - \frac{\pi^2}{3} \chi(n, \gamma) - \chi''(n, \gamma) - 2\Phi(n, \gamma) - 2\Phi(n, 1 - \gamma)$$

where

$$\chi(n, \gamma) = 2\psi(1) - \psi(\gamma + \frac{n}{2}) - \psi(1 - \gamma + \frac{n}{2})$$

$$\begin{aligned} \Phi(n, \gamma) &= \int_0^1 \frac{dt}{1+t} t^{\gamma-1+\frac{n}{2}} \left\{ \frac{\pi^2}{12} - \frac{1}{2} \psi' \left(\frac{n+1}{2} \right) - \text{Li}_2(t) - \text{Li}_2(-t) \right. \\ &\quad \left. - \left(\psi(n+1) - \psi(1) + \ln(1+t) + \sum_{k=1}^{\infty} \frac{(-t)^k}{k+n} \right) \ln t - \sum_{k=1}^{\infty} \frac{t^k}{(k+n)^2} [1 - (-1)^k] \right\} \end{aligned}$$

Agrees with $j \rightarrow 1$ asymptotics of 3-loop splitting functions

Vogt, Moch, Vermaseren,(2003)

Assembling NLO $F(\nu)$

Impact factor should not scale with energy $\Rightarrow a = \frac{x+y_+}{(x-y)^2}$ (analog of $\mu^2 = Q^2$ in DIS)

$$(x-y)^4 T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} = \frac{(x-y)^4}{\pi^2} \int d^2 z_1 d^2 z_2 \frac{(x+y_+)^{-2}}{\mathcal{Z}_1^2 \mathcal{Z}_2^2} \left\{ \hat{\mathcal{U}}^{\text{conf}} - \frac{\lambda}{4\pi^2} \int \frac{d^2 z_3 z_{12}^2}{z_{13}^2 z_{23}^2} \left[\ln \frac{x+y_+ z_{12}^2 e^{2\eta}}{(x-y)_\perp^2 z_{13}^2 z_{23}^2 \mathcal{Z}_3^2} - i\pi \right] [\hat{\mathcal{U}}^{\text{conf}}(z_1, z_3) + \hat{\mathcal{U}}^{\text{conf}}(z_2, z_3) - \hat{\mathcal{U}}^{\text{conf}}(z_1, z_2)] \right\}$$

The projection onto the conformal eigenfunctions $\left(\frac{z_{12}^2}{z_{10}^2 z_{20}^2} \right)^\gamma$ ($\gamma = \frac{1}{2} + i\nu$)

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The projection onto the conformal eigenfunctions $\left(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\right)^\gamma$ ($\gamma = \frac{1}{2} + i\nu$)

$$\int dz_1 dz_2 (x-y)^4 T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}(y)\} \left(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\right)^\gamma = \left(\frac{\kappa^2}{(2\kappa \cdot \zeta_0)^2}\right)^\gamma [I_{\text{LO}}^A(\gamma) + I_{\text{NLO}}^A(\gamma)] \hat{\mathcal{U}}(z_0, \gamma),$$

$$\hat{\mathcal{U}}(z_0, \gamma) = \int d^2 z_1 d^2 z_2 \left(\frac{z_{12}^2}{z_{10}^2 z_{20}^2}\right)^\gamma \hat{\mathcal{U}}(z_1, z_2)$$

$$I_{\text{LO}}^A(\gamma) = \frac{\Gamma^2(1-\gamma)}{\Gamma(2-2\gamma)} \Gamma(1+\gamma) \Gamma(2-\gamma)$$

$$I_{\text{NLO}}^A(\gamma) = \frac{\lambda}{16\pi^2} I_{\text{LO}}^A \left[\chi^2(\gamma) - \psi'(\gamma) - \psi'(1-\gamma) - 2 + \frac{\pi^2}{3} - F(\gamma) - F(1-\gamma) \right]$$

Assembling NLO $F(\nu)$

Similarly

$$\int dz_1 dz_2 (x' - y')^4 T\{\hat{O}(x') \hat{O}(y')\} \left(\frac{z_{12}^2}{z_{10}^2 z_{20}^2} \right)^{1-\gamma} = \left(\frac{\kappa^2}{(2\kappa \cdot \zeta_0)^2} \right)^{1-\gamma} [I_{\text{LO}}^A(\gamma) + I_{\text{NLO}}^A(\gamma)] \hat{\mathcal{V}}(z_0,$$

$$\hat{\mathcal{V}}(z_0, \gamma) = \int d^2 z_1 d^2 z_2 \left(\frac{z_{12}^2}{z_{10}^2 z_{20}^2} \right)^\gamma \hat{\mathcal{V}}(z_1, z_2)$$

$$I_{\text{LO}}^B(\gamma) = \frac{\Gamma^2(1+\gamma)}{\Gamma(2+2\gamma)} \Gamma(1+\gamma) \Gamma(2-\gamma)$$

$$I_{\text{NLO}}^B(\gamma) = \frac{\lambda}{16\pi^2} I_{\text{LO}}^B \left[\chi^2(\gamma) - \psi'(\gamma) - \psi'(1-\gamma) - 2 + \frac{\pi^2}{3} - F(\gamma) - F(1-\gamma) \right]$$

Assembling NLO $F(\nu)$

The last ingredient is the amplitude of scattering of two conformal dipoles
 $(\gamma \equiv \frac{1}{2} + i\nu)$

$$\langle \hat{\mathcal{U}}(z_0, \gamma) \hat{\mathcal{V}}(z'_0, \gamma) \rangle = \delta(\nu - \nu') \delta(z - z') R^{\frac{1}{2}\omega(\nu)} [A_{\text{LO}}(\gamma) + A_{\text{NLO}}(\gamma)]$$

$$A_{\text{LO}}(\gamma) = \frac{\Gamma(-\gamma)\Gamma(\gamma-1)}{\Gamma(1+\gamma)\Gamma(2-\gamma)}, \quad A_{\text{NLO}}(\gamma) = -\frac{\lambda}{4\pi^2} A_{\text{LO}} \left[\frac{\chi(\gamma)}{\gamma(1-\gamma)} - \frac{\pi^2}{3} \right]$$

The result is $F(\nu) = F_{\text{LO}}(\nu) + \lambda F_{\text{NLO}}(\nu) + O(\lambda^2)$

$$F_{\text{LO}}(\nu) = I_{\text{LO}}^A(\nu) A_{\text{LO}}(\nu) I_{\text{LO}}^B(\nu),$$

$$F_{\text{NLO}}(\nu) = I_{\text{NLO}}^A(\nu) A_{\text{LO}}(\nu) I_{\text{LO}}^B + I_{\text{LO}}^A(\nu) A_{\text{NLO}}(\nu) I_{\text{LO}}^B + I_{\text{NLO}}^A(\nu) A_{\text{LO}}(\nu) I_{\text{NLO}}^B(\nu)$$

NLO evolution of composite “conformal” dipoles in QCD

$$\begin{aligned}
 \frac{d}{d\eta} \text{tr}\{U_x U_y^\dagger\} &= \frac{\alpha_s}{2\pi^2} \int d^2 z \left([\text{tr}\{U_x U_z^\dagger\} \text{tr}\{U_z U_y^\dagger\} - N_c \text{tr}\{U_x U_y^\dagger\}] \right. \\
 &\times \frac{(x-y)^2}{X^2 Y^2} \left[1 + \frac{\alpha_s N_c}{4\pi} (b \ln(x-y)^2 \mu^2 + b \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} + \frac{67}{9} - \frac{\pi^2}{3}) \right] \\
 &+ \frac{\alpha_s}{4\pi^2} \int \frac{d^2 z'}{(z-z')^4} \left\{ \left[-2 + \frac{X'^2 Y^2 + Y'^2 X^2 - 4(x-y)^2(z-z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right] \right. \\
 &\times [\text{tr}\{U_x U_z^\dagger\} \text{tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} - (z' \rightarrow z)] \\
 &+ \frac{(x-y)^2(z-z')^2}{X^2 Y'^2} \left[2 \ln \frac{(x-y)^2(z-z')^2}{X'^2 Y^2} + \left(1 + \frac{(x-y)^2(z-z')^2}{X^2 Y'^2 - X'^2 Y^2} \right) \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right] \\
 &\times [\text{tr}\{U_x U_z^\dagger\} \text{tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z' \rightarrow z)] \left. \right\}
 \end{aligned}$$

$$b = \frac{11}{3}N_c - \frac{2}{3}n_f, \quad X \equiv x - z, \quad Y \equiv y - z, \quad X' \equiv x - z', \quad Y' \equiv y - z'$$

$K_{\text{NLO BK}}$ = Running coupling part + Conformal "non-analytic" (in j) part
 + Conformal analytic ($\mathcal{N} = 4$) part

Linearized $K_{\text{NLO BK}}$ reproduces the known result for the forward NLO BFKL kernel.

Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.

Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.
- The NLO BK kernel in for the evolution of conformal composite dipoles in $\mathcal{N} = 4$ SYM is Möbius invariant in the transverse plane.
- The NLO BK kernel agrees with NLO BFKL eigenvalues.
- The correlation function of four Z^2 operators is calculated at the NLO order.